Modified Description of Wave Motion in a Falling Film

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The presence of gravity-capillary waves at the surface of a viscous falling film has been studied theoretically by a more rigorous linear treatment than presently available.

The present analysis is based on the assumption of steady state periodic solutions of the complete Navier-Stokes equations. It provides methods of prediction, to a first order of approximation, for the wavelength, celerity, and wave number, in terms of the Weber number which emerges as the governing dimensionless group. This treatment includes as special cases the low Weber number analyses of Yih, Benjamin, Hanratty and Hershman, and Kapitza, as well as the high Weber number theory of Ishihara, Iwagaki, and Iwasa. Agreement with experimental observation is improved over that obtained from previous analyses.

The stream function of the system has been derived and the instantaneous streamlines have **been constructed for application to heat and mass transfer.**

The behavior of viscous liquid films under the influence of gravity and surface tension is a fundamental problem of great practical importance in such chemical engineering operations as distillation, gas absorption, and condensation of vapors, where heat and mass transfer are intimately connected to fluid motion. Meaningful heat and mass transfer studies can only be made after a realistic fluid mechanical model has been established. Recent extensive reviews by Norman (*22*), Dukler and Wicks (*7*), Levich (*20*), and Fulford (*8*) stress the importance of such a development. However, despite several hundreds of published investigations of falling films, very little has been ascertained on this extremely complex subject. The main sources of difficulty arise from the presence of gravity-capillary waves, which seem to defy observation and understanding, as can be seen from many contradictory findings compiled by Fulford (*8*).

Furthermore, the number of theoretical publications is surprisingly small. In these few works three approaches have been used to analyze the behavior of rippling films: search for periodic steady states, stability analysis, and semiempirical characterizations. The first approach is used in the theoretical analysis presented below. It provides a new and more general description, which includes all previous theories as special cases.

SIGNIFICANT RESULTS OF PRIOR WORK

The first major contribution to the analysis of falling film was Nusselt's solution (*23*) of the equation of motion reduced to

$$g + \nu \frac{\partial^2 x_x}{\partial y^2} = 0 \tag{1}$$

in terms of the coordinates of Figure 1, for steady rectilinear flow. His now famous results take the form:

$$v_x = \frac{3\, v_0}{h_0}\left(y - \frac{y^2}{2\, h_0}\right) \tag{2}$$

$$v_0 = \frac{1}{h_0}\int_0^{h_0} v_x dy = \frac{g\, h_0^2}{3\nu} \tag{3}$$

$$h_0 = \left(\frac{3\nu Q}{g}\right)^{\frac{1}{3}} \tag{4}$$

This theory is invalid when waves appear at the free surface, but it is widely used as a first approximate description. As we shall see, it is also useful in the development of the more elaborate analyses discussed below.

Surface waves are normally encountered at $N_{Re} > 20$. Since they considerably modify film behavior, for example, greatly increasing mass transfer rates, recent investigations have been centered on them.

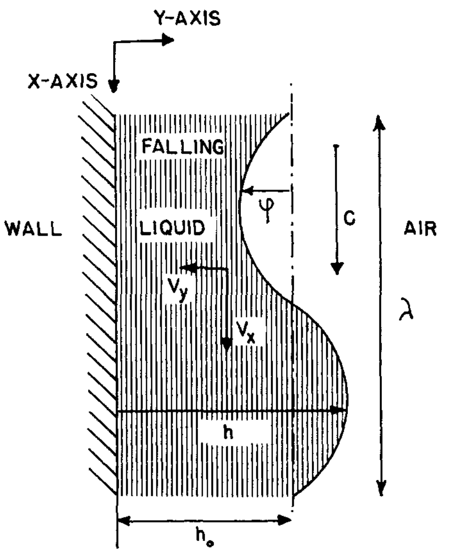

Fig. 1. Wave on a falling liquid film.

Search for Periodic Steady State Laminar Solutions

Kapitza (*13*) was the first to attempt a description of a wavy flow by postulating an oscillatory steady state solution of the equations of motion. This approach has achieved a considerable success and provides the most complete description of the waves available. Since this theory has been described in detail by Levich (*20*), it will only be useful to recall some of its important results and to point out the differences with the present analysis, which is essentially an extension of Kapitza's work.

Kapitza (*13*) and others (*5, 20*) considered the equations of motion and boundary conditions for thin films and they took into account surface tension effects. Two major approximations, which will be corrected in this treatment were: $\nu\ (\partial^2 v_x)/(\partial x^2)$ was neglected, as in a boundary-layer approximation, and the variation of the surface coordinate h with the longitudinal coordinate x, was neglected in most of the derivations.* These workers developed as a first approximation, a linearized treatment yielding the differential equation:

$$\frac{\sigma h_0}{\rho}\frac{d^3\phi}{dx^3} + (c - v_0)\left(c - \frac{9}{10}v_0\right)\frac{d\phi}{dx} - \frac{3\nu}{h_0^2}(c - 3v_0)\,\phi + g - \frac{3\nu\, v_0}{h_0^2} = 0 \quad (5)$$

where ϕ is the free surface deformation function defined by

$$h = h_0\,(1 + \phi) \quad (6)$$

and is the steady state periodic solution of Equation (5) given by

$$\phi = C_1 \sin\frac{2\pi}{\lambda}(x - ct) \quad (7)$$

Other important results include the relation between h_0 and v_0, which is found to be the Nusselt relation given by Equation (3), and the relation between the wave celerity c and v_0, which is

$$c/v_0 = \alpha = 3 \quad (8)$$

α is found to be a constant with respect to the flow and physical properties. Expressions for the wavelength are given in Table 1. The results are compared with experimental values in Figures 2, 3, and 5. The agreement is good at low flow rates but at higher regimes of flow, they fail in that, for increasing rate of flow, the wavelength does not decrease as predicted but passes by a minimum then increases, and the dimensionless wave celerity α is not constant as predicted, but decreases. By determining the minimum viscous energy dissipation, the dimensionless amplitude C_1 was found to be 0.46. This is a reasonable but unproven criterion.

Kapitza obtained at a second order of approximation a nonlinear treatment, to which it is not possible to subscribe because of a lack of justification. In particular he used in his calculations results from the minimization of viscous energy dissipation. Such a minimum has not yet been proven to correspond to the state of the system. However, the nonlinear predictions are shown in Figures 2, 3, 5. In the following it will be, unless otherwise specified, referred only to Kapitza's linear treatment.

A quite different periodic solution was obtained by Ishihara, Iwagaki, and Iwasa (*11*). These workers solved the hydraulic equation of motion for unsteady flow in open channel with a constant inclination. Neglecting the surface tension effects, they obtained a value of the dimensionless celerity given by

$$\alpha = \frac{6}{5} + \sqrt{\frac{6}{25} + \frac{1}{S\,N_{Re}}} \quad (9)$$

where S is the slope of the channel. As we shall see both this result and the Kapitza expression for α are included as special cases in the more general treatment given below. Some defects of Kapitza's treatment have recently been pointed out by Kasimov and Zigmund (*16*), namely, the constancy of h with x. However, these authors failed to produce a self-consistent alternate in that their result of growing waves is not compatible with their assumption of periodic steady state.

Hydrodynamic Instability Theory

The stability of Nusselt's solution has been investigated in terms of the linearized theory of small perturbations by Yih (*29*), Benjamin (*2*), Hanratty and Hershman (*9*), and Whitaker (*28*). By postulating that waves are generated by infinitesimal periodic disturbances, and that the most likely to be seen are the fastest growing ones, this theory can be used to obtain a partial description of the wave motion. Such analyses are, however, of limited value for describing waves of finite amplitude. Furthermore, Hunt (*10*) points out that some flows stable under linear perturbation are unstable under nonlinear perturbations. Predictions of this theory come remarkably close to Kapitza's linear treatment, as can be seen in Table 1 and in Figures 2, 3, and 5, where they are compared to other theories and experimental data.

Discussion of the Assumption of Periodic Steady State

A number of researchers have analyzed wavy films from a statistical standpoint (*24*) or used the semiempirical turbulent model developed for flow in closed conduit (*6, 18*), postulating a random or turbulent behavior. These works are not detailed further here, since only the ap-

* In his own treatment Kapitza had also neglected the term $v_y(\partial v_x)/(\partial y)$ but his omission was later corrected by Bushmanov (*5*).

TABLE 1. VALUES OF DIMENSIONLESS CELERITY AND WAVELENGTH GIVEN BY VARIOUS THEORIES

Treatment	Linear	Nonlinear
Kapitza (*13*)	$\alpha = 3$ $A = 1.29$	$\alpha = 2.4$ $A = 2.39$
Bushmanov (*5*) Levich (*20*)	$\alpha = 3$ $A = 1.024$	$\alpha = 2.4$ $A = 2.11$
Yih (*29*) Benjamin (*2*) Hanratty and Hershman (*9*)	$\alpha = 3$ $A = 0.705$	
Ishihara, Iwagaki, and Iwasa (*11*) (no surface tension)	$\alpha = \frac{6}{5} + \sqrt{\frac{6}{25}}$	

Present linear treatment:

$$\alpha = \alpha(N_{We})$$

$$A = \frac{\sqrt{12}}{\sqrt{\alpha^2 - \frac{12}{5}\alpha + \frac{6}{5}}}$$

$N_{We} \to 0$	$\alpha \to 3$	$A \to 2$
$N_{We} \to \infty$	$\alpha \to \frac{6}{5} + \sqrt{\frac{6}{25}}$	$A \to \infty$
$\alpha = \frac{c}{v_0}$	$N_\lambda = A\,N_{Re}^{-1/2}$	

parently laminar range in which regular waves occur is being considered. Working on a 3-in. I.D. wetted-wall column, the authors (*21*) have been able to prove experimentally that random perturbations applied to the liquid input line generate random waves along the whole length of the column. On the other hand, when special precautions were taken to filter out such perturbations, coming, for example, from the drainage line of the constant-head feeding tank, remarkably periodic waves have been observed, at least near the top of the column. This fact and the sizable departure between existing predictions and observations suggest a more rigorous search for periodic solutions of the fluid equations of motion. This is the purpose of the present work. It can be added, incidentally, that published data on waves can be considered as unreliable if in the experiments care has not been taken to absorb and damp external perturbations of the liquid line.

PRESENT THEORY

The present theory is based on the search for an oscillatory steady state solution of the Navier-Stokes equations.

Equations and Boundary Conditions

The development of this theory begins with the equations of continuity and motion for two-dimensional vertical flow of an incompressible Newtonian fluid of constant viscosity. The present work differs basically from that of Kapitza (*13*) and others (*5, 20*) in that the complete x component equation of motion is considered [the term $(\partial^2 v_x)/(\partial x^2)$ is not neglected], the complete y component equation of motion is considered, and the variation of h with x is explicitly taken into account.

The equation of continuity

$$(\partial v_x)/(\partial x) + (\partial v_y)/(\partial y) = 0 \tag{10}$$

and the Navier-Stokes equations

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial x^2} + \nu \frac{\partial^2 v_x}{\partial y^2} + g - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{11}$$

$$\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \nu \frac{\partial^2 v_y}{\partial x^2} + \nu \frac{\partial^2 v_y}{\partial y^2} - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{12}$$

are to be solved with addition of the boundary conditions.

$$\frac{\partial v_x}{\partial y} = 0 \quad \text{at } y = h(x,t) \quad \text{for all } x \text{ and } t \tag{13}$$

$$v_x = v_y = 0 \quad \text{at } y = 0 \quad \text{for all } x \text{ and } t \tag{14}$$

$$p = p_\sigma = \sigma \frac{\partial^2 h}{\partial x^2} \quad \text{at } y = h(x,t) \tag{15}$$

Equation (15) is an approximation to the third order in power of $(\partial h)/(\partial x)$ of the capillary pressure due to the curvature of the surface. In addition, the boundary-layer approximation of hydrostatic equilibrium across the film will be assumed to be valid, hence [following Kapitza (*13*) and Levich (*20*)]

$$(\partial p)/(\partial y) = 0 \tag{16}$$

To solve these equations, the Nusselt result will be used locally in the following form for the x component of velocity (as in references *13* and *20*):

$$v_x = \frac{3\bar{v}(x,t)}{h(x,t)}\left(y - \frac{y^2}{2h(x,t)}\right) \tag{17}$$

$\bar{v}$ and h are to be determined as functions of x and t and are to be treated as such throughout all the calculations. From here on the solution differs from Kapitza's and Levich's.

The y component of velocity is derived from Equation (17) through the use of the equation of continuity (10). The result is

$$v_y = -\frac{3}{h(x,t)}\frac{\partial \bar{v}}{\partial x}\left(\frac{y^2}{2} - \frac{y^3}{6h(x,t)}\right) + \frac{3v(x,t)}{h^2(x,t)}\frac{\partial h(x,t)}{\partial x}\left(\frac{y^2}{2} - \frac{y^3}{3h(x,t)}\right) \tag{18}$$

More complex expressions v_x have been proposed (*15*). However, even the present choice generates fairly complicated calculations and its possibilities are far from being exhausted. In fact, this simple expression has some remarkable properties. As it stands it reduces to the Nusselt expression for h constant, verifies the boundary conditions without presupposing any shape of the surface, satisfies the requirement of no net flow in the y direction, and does not require any a priori assumption on the wave amplitude in that the average of the flow rate over a wave length is independent of the amplitude or wavelength. Furthermore, as will be seen later expressions, (17) and (18) yield streamlines comparable with those for other well-known wave motion, asymptotic results consistent with other theories, and results closer to data than previous theories.

Linearization of the Navier-Stokes Equations

After insertion of expressions (17) and (18) into Equations (11) and (12), following Kapitza (*13*) and Levich (*19*), one averages these equations over the cross section by integrating them with respect to y. Then, since only solutions periodic in x and t are to be considered, the problem can be expressed in terms of the single variable

$$x = x - ct$$

This relation introduces the wave celerity c. After lengthy but straightforward calculations, Equation (11) can be written

$$-c\frac{\partial \bar{v}}{\partial x} + \frac{1}{2}c\frac{\bar{v}}{h}\frac{\partial h}{\partial x} + \frac{9}{10}\bar{v}\frac{\partial \bar{v}}{\partial x} - \frac{3}{10}\frac{\bar{v}^2}{h}\frac{\partial h}{\partial x} = \frac{\sigma}{\rho}\frac{\partial^3 h}{\partial x^3} + \nu\frac{\partial^2 \bar{v}}{\partial x^2} - \nu\frac{\bar{v}}{2h}\frac{\partial^2 h}{\partial x^2} + g - \frac{3\nu\bar{v}}{h^2} \tag{19}$$

In the same fashion, Equation (12) becomes

$$\frac{3}{8}ch\frac{\partial^2 \bar{v}}{\partial x^2} - c\frac{\bar{v}}{4}\frac{\partial^2 h}{\partial x^2} - \frac{1}{2}\bar{v}h\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{13}{40}\bar{v}^2\frac{\partial^2 h}{\partial x^2} = -\frac{3}{8}\nu\frac{\partial^3 \bar{v}}{\partial x^3}h + \frac{1}{4}\nu\bar{v}\frac{\partial^3 h}{\partial x^3} - \frac{3}{2h}\nu\frac{\partial \bar{v}}{\partial x} \tag{20}$$

In Equations (19) and (20) products of derivatives have been omitted as they would introduce nonlinear terms in the following development.

To solve Equations (19) and (20) a relation is needed between the two dependent variables $\bar{v}$ and h. This is provided by the macroscopic mass balance which may be written in the form

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\int_0^h v_x dy = -\frac{\partial}{\partial x}(vh) \tag{21}$$

In addition, it will prove mathematically convenient to replace h with a new variable $\phi(x)$, of mean value zero, defined by Equation (6).

$$h = h_0 (1 + \phi)$$

By integrating Equation (21) over a wavelength and by using the celerity c, one obtains an important relation between $\bar{v}$ and ϕ, given by Kapitza (13) and Levich (19):

$$v = \frac{c\phi + v_0}{1 + \phi} \tag{22}$$

which presupposes only a periodic shape of the surface. If one neglects quadratic and higher powers of ϕ, assumed small, as well as its derivatives, it follows that

$$\bar{v} \approx c\phi + v_0 \tag{23}$$

The variable ϕ, therefore, simplifies the task of eliminating $\bar{v}$ from Equations (19) and (20)

Equations (6) and (23) are substituted into (19) and (20) to give

$$\frac{\sigma h_0}{\rho}\frac{\partial^3\phi}{\partial x^3} + \nu\left(c - \frac{3}{2}v_0\right)\frac{\partial^2\phi}{\partial x^2} + \left(c^2 - \frac{12}{5}cv_0 + \frac{6}{5}v_0^2\right)\frac{\partial\phi}{\partial x} - \frac{3\nu(c - 3v_0)}{h_0^2}\phi - \frac{3\nu v_0}{h_0^2} + g = 0 \tag{24}$$

and

$$\frac{\nu h_0}{8}\frac{\partial^3\phi}{\partial x^3}(-3c + 5v_0) - \frac{h_0}{40}\frac{\partial^2\phi}{\partial x^2}(15\,c^2 - 45\,cv_0 + 33\,v_0^2) - \frac{3\nu}{2h_0}(c - v_0)\frac{\partial\phi}{\partial x} = 0 \tag{25}$$

These two equations with the conditions (13), (14), (15), and (16) constitute the film description. A comparison between Equations (24) and (25) and Equation (5) points out the differences between the present treatment and Kapitza's, as well as the heavy repercussions of Kapitza's approximations on h and $(\partial^2 v_x)/(\partial x^2)$.

Existence of Periodic Solutions

x *Component of Equation of Motion.* Equation (24) is a third-order linear differential equation with constant coefficients. We now seek a periodic solution of this equation for which the wave shape and amplitude do not change from one period to the next, that is, we seek solutions describing a periodic steady state.

Since the mean value of ϕ is zero, the constant term of Equation (24) must vanish. This result, obtained by Kapitza (13), fixes the mean value of the film thickness h_0 at the Nusselt value given by Equation (3)

$$g = (3\nu v_0)/(h_0^2)$$

In order to write Equation (24) in a dimensionless form this relation will be used to express all coefficients in terms of h_0 only.

Without any loss of generality one can substitute for c the expression

$$c = \alpha\, v_0 \tag{26}$$

where the dimensionless celerity α is to be determined.

By introducing a wavelength λ and a dimensionless length X such that

$$X = 2\pi\frac{x}{\lambda} \tag{27}$$

then

$$\phi' = \frac{2\pi}{\lambda}\frac{\partial\phi}{\partial X}, \phi'' = \ldots \tag{28}$$

By considering, in addition, a dimensionless wavelength

$$N_\lambda = \frac{\lambda}{2\pi}\sqrt{\frac{g}{\gamma}} \tag{29}$$

and a dimensionless wave number

$$N_W = \frac{2\pi h_0}{\lambda} \tag{30}$$

the x component equation of motion given by Equation (24) can be written in the equivalent dimensionless form

$$\frac{N_W}{N_\lambda^2}\phi''' + \frac{1}{3}N_W^2\left(\alpha - \frac{3}{2}\right)\phi'' + N_W N_{Fr}^2\left(\alpha^2 - \frac{12}{5}\alpha + \frac{6}{5}\right)\phi' - (\alpha - 3)\,\phi = 0 \tag{31}$$

where the Froude number is given by

$$N_{Fr} = \frac{\sqrt{g}\,h_0^{3/2}}{3\nu} = \frac{v_0}{\sqrt{g\,h_0}} \tag{32}$$

Equation (31) has a periodic steady solution only if the dimensionless celerity α and the Weber number of the system satisfy the unique relation

$$N_{We} = \frac{3(3 - \alpha)}{\left(\alpha - \frac{3}{2}\right)\left(\alpha^2 - \frac{12\alpha}{5} + \frac{6}{5}\right)} \tag{33}$$

$$N_{We} > 0 \quad \text{if} \quad \frac{6}{5} + \sqrt{\frac{6}{25}} < \alpha < 3 \tag{34}$$

where

$$N_{We} = \frac{v_0^2\,h_0}{\gamma} = \frac{g^2\,h_0^5}{9\nu^2\,\gamma} \tag{34}$$

It can be noticed that from the present treatment, the Weber number evolves as the determining dimensionless group for falling film waves. So far in the literature only the Reynolds and the Froude numbers had been related theoretically to this problem.

The periodic solution consistent with Equation (33) is

$$\phi = C_1 \sin X \tag{35}$$

$$= C_1 \sin\frac{2\pi}{\lambda}(x - ct) \tag{36}$$

at

$$X = 0, \quad \phi = 0, \quad \bar{v} = v_0, \quad h = h_0$$

C_1 is an integration constant representing the dimensionless amplitude. Its estimation, which cannot be made from Equation (31), will be discussed later.

The wavelength λ is given by

$$\lambda = \frac{2\pi\,h_0\sqrt{\alpha - 3/2}}{\sqrt{3(3 - \alpha)}} = 2\pi\frac{\sqrt{\lambda/g}}{\sqrt{\alpha^2 - \frac{12}{5} + \frac{6}{5}}}\frac{3\nu}{g^{1/2}\,h_0^{3/2}} \tag{37}$$

$$\lambda > 0 \quad \text{if} \quad \frac{6}{5} + \sqrt{\frac{6}{25}} < \alpha < 3 \tag{38}$$

or in dimensionless form

$$1 = \frac{3(3 - \alpha)}{(\alpha - 3/2)}\frac{1}{N_W^2} = N_{Fr}^2\,N_\lambda\left(\alpha^2 - \frac{12}{5}\alpha + \frac{6}{5}\right) \tag{38}$$

The celerity c is given by $c = \alpha\,(N_{We})v_0$.

y *Component Equation of Motion.* By introducing the Nusselt expression for v_0 given by Equation (3) and the dimensionless groups mentioned above, the y component equation of motion Equation (25) can be written in the equivalent dimensionless form

$$\frac{1}{3} N_W{}^2 \left(\alpha - \frac{5}{3}\right)\phi''' + N_W N_{Fr}{}^2 \left(\alpha^2 - 3\alpha + \frac{11}{5}\right)\phi'' + \frac{4}{3}(\alpha - 1)\,\phi' = 0 \quad (39)$$

The only value of the dimensionless celerity α giving a steady state periodic solution of Equation (39) is given by the greater root of the equation

$$\alpha^2 - 3\alpha^2 + \frac{11}{5} = 0 \quad (40)$$

$$\alpha = \frac{3 + \sqrt{1/5}}{2} = 1.72 = \text{constant} \quad (41)$$

The corresponding wavelength is

$$\lambda = 2\pi\, h_0 \sqrt{\frac{(\alpha - 5/3)}{4(\alpha - 1)}} \quad (42)$$

$$\lambda > 0 \text{ if } \alpha > \frac{5}{3}$$

or in dimensionless form

$$1 = \frac{4(\alpha - 1)}{(\alpha - 5/3)} \frac{1}{N_W{}^2} \quad (43)$$

Simultaneous Solutions of Equations of Motion. There are no periodic solutions satisfying simultaneously Equations (31) and (39), because it is not possible to find a value of the dimensionless celerity α which satisfies simultaneously three out of four of the following requirements for periodic solutions of equal wavelength:

$$\frac{3(3-\alpha)}{(\alpha - 3/2)} \frac{1}{\left(\alpha^2 - \frac{12}{5}\alpha + \frac{6}{5}\right)} = N_{We} \quad (33)$$

$$\frac{4(\alpha-1)}{(\alpha - 5/3)} \frac{1}{\left(\alpha^2 - \frac{12}{5}\alpha + \frac{6}{5}\right)} = N_{We} \quad (44)$$

$$\alpha^2 - 3\alpha + \frac{11}{5} = 0 \quad (45)$$

$$\alpha^2 - \frac{24}{7}\alpha + 3 = 0 \quad (46)$$

This is not surprising since, as shown by Levich (*19*) by a consideration of orders of magnitude, the y equation of motion is of second order with respect to the x component. This is a consequence of the particular geometry of the thin film in which

$$v_x >> v_y$$

Hence, the linearized solution of (31) should not be expected to verify (39). It is reasonable to anticipate that only a higher order treatment will satisfy both equations.

RESULTS OF THE PRESENT THEORY

Wave Description

Since Equation (31) has been shown to be the determining equation of the problem, properties of its solution can be studied. Results of the above analysis are shown graphically and will be discussed briefly.

Wave Celerity. The theoretical wave celerity is plotted in Figure 2 against the Weber number, together with experimental data (*21*) and compared with other theoretical and experimental results (*13, 14, 2, 5, 9*). The agreement can be considered as good, with experimental values from 15% to 50% lower than the theoretical ones.

Wavelength. From expression (37) the dimensionless wavelength N_λ comes out as a function of the Weber number, and can be written as

$$N_\lambda = \left(\alpha^2 - \frac{12}{5}\alpha + \frac{6}{5}\right)^{-1/2} N_{Fr}{}^{-1} \quad (47)$$

When α tends toward 3, the limiting form of Equation (47) is

$$N_\lambda = 2.00\ N_{Re}{}^{-1/2} \quad (48)$$

These relations, plotted in Figure 3, show the variation of N_λ against the Weber number. It can be seen that the wavelength passes by a minimum and becomes very large for small and large Weber numbers. Such a behavior, in contradiction with everdecreasing wavelength predictions (*13, 2, 29*), has been observed experimentally by Fulford (*8*) and by Massot and Irani (*21*) and is borne out by the Kapitza data (*14*). However, experimental results seem consistently higher than theoretical predictions. This fact will be considered later in the discussions of linearity and of the determination of the amplitude.

Wave Number. From Equations (37) or (38) the wave number N_W can be obtained. It has been plotted in Figure 4 against the dimensionless wavelength N_λ. The path of the system for increasing Weber number goes through three regions related by Robertson (*27*) to different known types of waves, namely; shallow water waves, capillary waves, and gravity or deep water waves. The transition between gravity and capillary waves is usually set in the literature at $N_\lambda = 1$, because this value corresponds

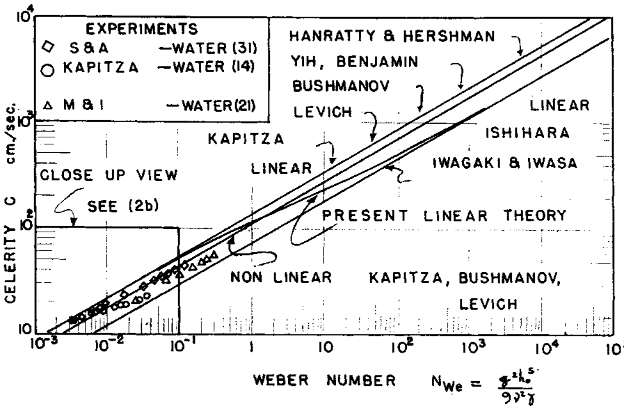

Fig. 2a. Comparison between various theoretical predictions and experiments for the variation of the celerity as a function of the Weber number. For close-up view, see Figure 2b.

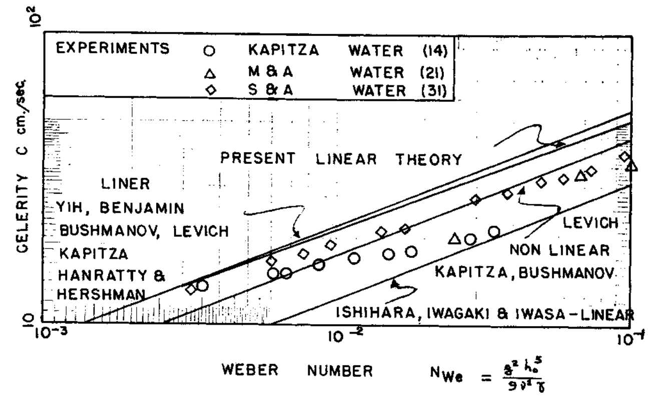

Fig. 2b. Close-up view of Figure 2a. Comparison between various predictions and observations for the celerity as a function of the Weber number.

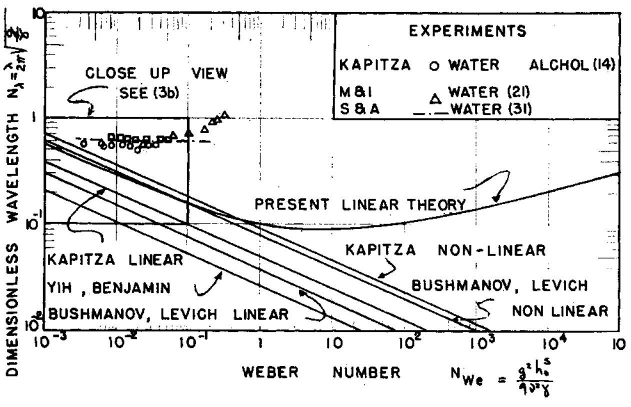

Fig. 3a. Comparison between various theories and experimental data for the variation of dimensionless wavelength N_λ in terms of the Weber number. For close-up view, see Figure 3b.

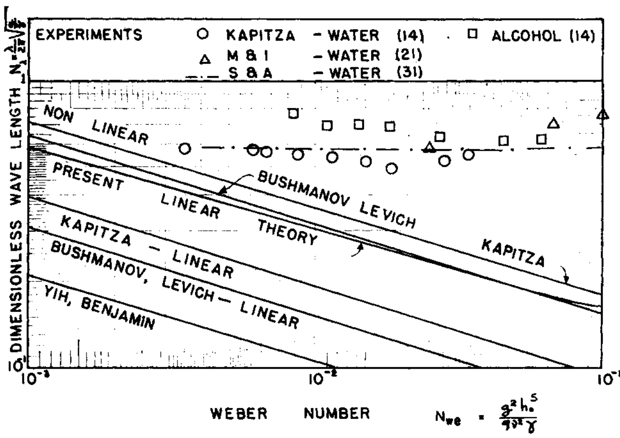

Fig. 3b. Close-up view of Figure 3a. Comparison between various theories and observations for the variation of N_λ vs. N_{We}.

to the minimum of celerity of an ideal fluid. The system can be seen in Figure 4 to cross some uncharted area. This plot illustrates the complexity of falling film waves. There is qualitative agreement between the present theory and observations.

Dimensionless Wave Celerity. A comparison between the existing theories and results appears in Figure 5, representing the variation of the dimensionless celerity α with the Weber number. It can be seen that the present theory includes Kapitza's (*13*), Yih's (*29*), Benjamin's (*2*), and Hanratty and Hershman's (*9*) theoretical results as the limiting case for low Weber number and Ishihara, Iwagaki, and Iwasa's (*11*) prediction as the limiting case for high Weber number. Furthermore, at medium Weber number, where other theories fail, there is qualitative agreement between experimental results and the present preditcions. It is remarable that all experimental observations of α lie between 3 and 1.7, which is the predicted range. The departure between theory and experiments can be due to nonlinear effects, discussed later. On the other hand, it may result from inadequate experimental technique, in particular, failure to maintain cleanliness of liquid surfaces, or to other complex secondary effects of as yet unknown nature. The major discrepancy between the Kapitza (*14*) and Fulford (*8*) results points to such a possibility.

Asymptotic Cases

The general nature of the present treatment is further emphasized by considerations of the asymptotic forms of Equation (31). To that end it is helpful to write Equation (31) in the form:

$$\frac{(2\pi)^3}{\lambda^3}\frac{\lambda h_0}{g}\phi''' + \frac{(2\pi)^2}{\lambda^2}\frac{h_0^2}{3}\left(\alpha - \frac{3}{2}\right)\phi'' + \frac{2\pi}{\lambda}\frac{gh_0^4}{9\nu^2}\left(\alpha^2 - \frac{12}{5}\alpha + \frac{6}{5}\right)\phi' - (\alpha - 3)\phi = 0 \quad (49)$$

The asymptotic behavior of the present theory seen in Figures 2, 3, 4, and 5 can be established mathematically.

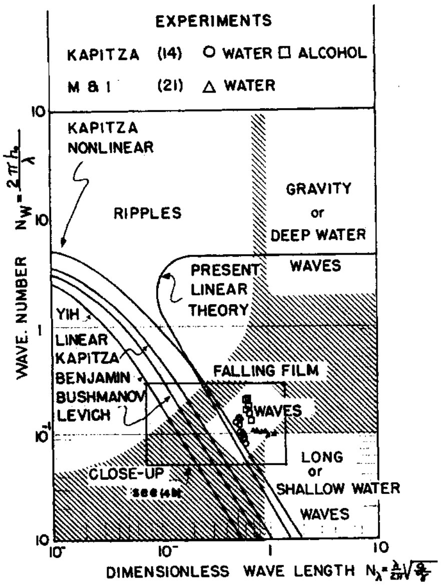

Fig. 4a. Comparison between theoretical falling film waves with known types of waves and experimental observations. For close-up view, see Figure 4b.

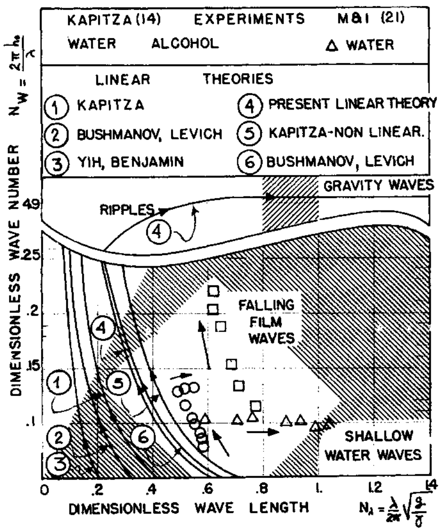

Fig. 4b. Close-up view of Figure 4a. Comparison between falling film waves as described by various theories, with known types of waves and experimental observations on the dimensionless chart N_λ vs. N_w. The arrows indicate the direction of increasing Weber number.

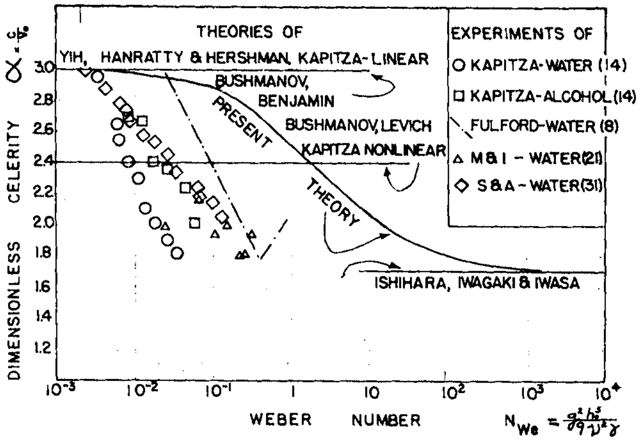

Fig. 5. Dimensionless celerity α as a function of the Weber number for existing theories, the present theory, and experimental observations.

Low Weber Number, Thin Films. Equation (49) reduces to

$$(\alpha - 3)\phi = 0 \tag{50}$$

A periodic solution can verify this expression only if $\alpha = 3$, which is the value predicted by the theories of Kapitza (*13*), Benjamin (*2*), Yih (*29*), Hanratty and Hershman (*9*). Hence, these theories are asymptotic to the present solution at low Weber number.

High Weber Number, Thick Films. The dominant term of Equation (49) is

$$\frac{2\,\pi\, g h_0^4}{9\nu^2\,\lambda}\left(\alpha^2 - \frac{12}{5}\,\alpha + \frac{6}{5}\right)\phi' = 0 \tag{51}$$

Here a periodic solution is possible only if

$$\alpha = \frac{6}{5} + \sqrt{\frac{6}{25}} = 1.689 \tag{52}$$

(The negative root is not acceptable because it would yield a negative value for the wavelength.) Expression (52) is the limiting value of the Ishihara, Iwagaki, and Iwasa (*11*) treatment given in Equation (9) for large Weber number. Their analysis is, therefore, asymptotic to the present one at high Weber number.

Low Wave Number. When N_W is small, Equation (49) reduces again to Equation (50). This is the case of long waves. The asymptote is again given by the Yih, Benjamin, Hanratty, and Hershman and Kapitza value $\alpha = 3$.

Thick Films with Long Waves. Equation (49) again reduces to Equation (51) and the Ishihara, Iwagaki, and Iwasa value is attained as limiting case.

No Surface Tension—Influence of Surface Active Agents. The Weber number becomes infinite. The limiting Ishihara, Iwagaki, and Iwasa value is reached again. Waves become very long and should not be visible. The present theory can thus provide a tentative explanation of the stabilization of a film by a surface active agent. The lowering of surface tension by the wetting agent increases the Weber number of the system and hence increases the wavelength of the waves so that they practically disappear. This argument appears to be in agreement with the experimental results of Brötz (*4*) and Jackson (*12*), who found little effect of the addition of surface active agent. Fulford (*8*) suggests that the cause could be that in these experiments wavelengths were long and hence were less likely to be affected by surface tension forces.

Theoretically this hypothesis is in agreement with Yih's treatment (*30*) of very small waves. From the hydrodynamic instability theory, Yih admits that very small waves are only compatible with zero surface tension.

Expression for the Velocities

According to expressions (17), (18), and (36) the velocity component v_x can be written, retaining only first-order terms

$$v_x = 3v_0\left(1 + \alpha\, C_1 \sin\frac{2\pi}{\lambda}(x - ct)\right)\left(\frac{y}{h} - \frac{y^2}{2h^2}\right) \tag{53}$$

Accordingly

$$v_y = -\,3v_0 h_0(\alpha - 1)\,\frac{2\pi}{\lambda} C_1 \cos\frac{2\pi}{\lambda}(x - ct)\left(\frac{y^2}{2h^2} - \frac{y^3}{6h^3}\right) + 3\,v_0 h_0 \frac{2\pi}{\lambda} C_1 \cos\frac{2\pi}{\lambda}(x - ct)\left(\frac{y^2}{2h^2} - \frac{y^3}{3h^3}\right) \tag{54}$$

with

$$h = h_0\left(1 + C_1 \sin\frac{2\pi}{\lambda}(x - ct)\right) \tag{55}$$

By introducing the dimensionless complete wave number

$$N_{cw} = \frac{2\pi\, h_0\, C_1}{\lambda} \tag{56}$$

the dynamic wave number

$$N_{dw} = \frac{2\pi\, h_0\, C_1\, \alpha}{\lambda} \tag{57}$$

and the celerity-amplitude number

$$N_{ca} = \alpha\, C_1 \tag{58}$$

the dimensionless velocities

$$N_{vx} = \frac{2}{3}\,\frac{v_x}{v_0} \tag{59}$$

$$N_{vy} = \frac{2}{3}\,\frac{v_y}{v_0} \tag{60}$$

can be written with Equations (53) and (54) and the dimensionless transverse coordinate $\delta = y/h$

$$N_{vx} = \left(1 + N_{ca} \sin\frac{2\pi}{\lambda}(x - ct)\right)(2\delta - \delta^2) \tag{61}$$

Accordingly

$$N_{vy} = \cos\frac{2\pi}{\lambda}(x - ct)\left(N_{cw}(2\delta^2 - \delta^3) - N_{dw}\left(\delta^2 - \frac{\delta^3}{3}\right)\right) \tag{62}$$

Discussion of the Linear Approximation

It is necessary to verify a posteriori if Equation (36) is consistent with the assumptions of linearity. Even when ϕ is small the derivatives

$$\phi' = \frac{2\pi}{\lambda} C_1 \sin\frac{2\pi}{\lambda} x,\ \phi'' = \ldots$$

can become very large when the wavelength goes to zero, and nonlinear terms, products of the derivatives, can become important. Fortunately, the present treatment excludes waves of zero wavelength. Hence, solution (36) is acceptable for small and large Weber numbers corresponding to large wavelength, while previous analyses (*13, 2*) are limited to small Weber numbers only or high Weber numbers only (*11*).

Even for medium Weber numbers, nonlinear effects should not be too large, since the minimum wavelength is not very small, but they can be expected to influence the results. Hence, departures between experiments and linear theory in this range should not be surprising. However, good predictions of Kapitza's (*13, 5, 20*) nonlinear theory appear accidental. Figure 5 shows that it does not reduce to the linear theory.

VISCOUS DISSIPATION

It remains to determine the wave amplitude C_1. Kapitza (*13*) has suggested that the correct value of this quantity is that which provides the least viscous dissipation of mechanical energy. No proof of this assertion has been established; hence, Kapitza's result is questionable. However, since Kapitza minimized only a simplified expression of the viscous dissipation, namely $\mu\left(\frac{\partial v_x}{\partial y}\right)^2$, it is interesting to find out if it is possible to find a minimum to the exact expression of the viscous energy dissipation. From (3) we may write

$$\Phi_v = 2\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2\right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]^2 - \frac{3}{2}\left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right]^2 \quad (63)$$

For an incompressible fluid, application of Equation (10) gives

$$\Phi_v = 4\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 \quad (64)$$

where Φ_v is the rate of viscous dissipation of mechanical energy per unit volume. If Φ_v is integrated over a cross section one obtains

$$\frac{h_0}{3v_0^2}\bar{\Phi}_v = \frac{3}{(1+\phi)^3}\left[\frac{4N_W^2}{60}C_1^2\cos^2\beta_x\right.$$
$$[8A^2 + 2B^2 - 7AB] + \frac{B^2}{3} + N_W^2\left[\frac{BAC}{15} + \frac{BD}{20}\right]$$
$$\left. + N_W^4\left[\frac{A^2C^2}{4}\frac{11}{420} + ACD\frac{41}{1260} + \frac{D^2}{9}\frac{3}{140}\right]\right] \quad (65)$$

with $A = \alpha - 1$, $B = \alpha\phi + 1$, $C = -\phi(1+\phi) - 2C_1^2\cos^2\beta x$, $D = -B\phi(1+\phi)$, $N_W = 2\pi\frac{h_0}{\lambda}$, $\beta = \frac{2\pi}{\lambda}$, $\phi = C_1\sin\beta x$.

For the Nusselt regime Φ_v is integrated over y to give

$$\bar{\Phi}_{VN} = \frac{3v_0^2}{h_0} \quad (66)$$

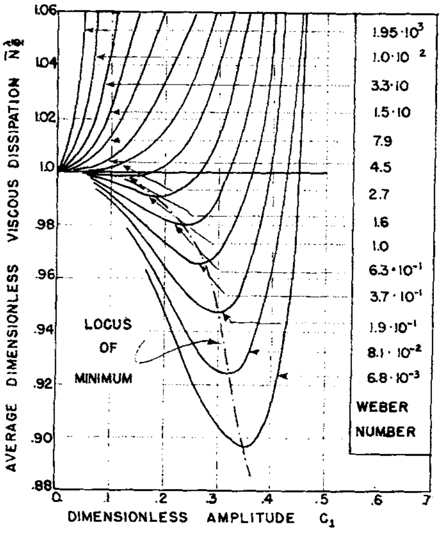

Fig. 6. Variation of $\bar{N}_\Phi^\lambda$ vs. C_1.

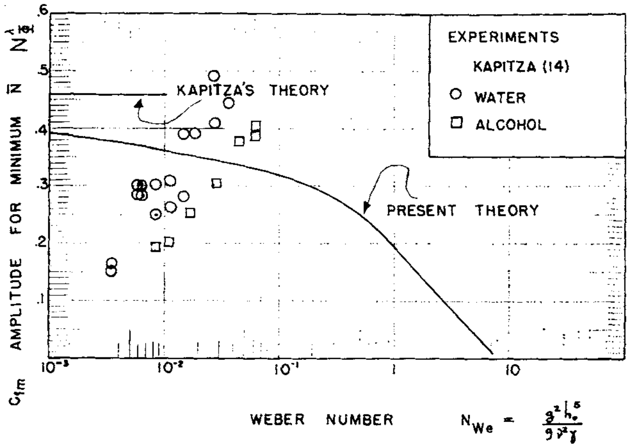

Fig. 7. Dimensionless amplitude C_{1m} corresponding to a minimum of $\bar{N}_\Phi^\lambda$, as a function of the Weber number compared with observations of C_1.

A dimensionless energy dissipation can be defined as

$$N_\Phi = \frac{\bar{\Phi}_v}{\bar{\Phi}_{VN}} \quad (67)$$

$$N_\Phi \to 1 \text{ when } N_{We} \to 0 \quad (68)$$

An average value of N_Φ over a wavelength has been obtained because of the complexity of its expression by numerical integration with the Gaussian-Legendre quadrature with fifteen weighted increments (*17*). The results are shown in Figure 6. It can be seen that

$$\bar{N}_\Phi^\lambda = \frac{1}{\lambda}\int_0^\lambda N_\Phi dx \quad (69)$$

for a given value of the Weber number can pass by a minimum when the dimensionless amplitude C_1 varies from 0 to 1. C_{1m}, the value of C_1 correspondingly to this minimum appears to be a decreasing function of the Weber number, asymptotic to the Kapitza value of 0.46 for zero Weber number. C_{1m} is compared with experimental values of wave amplitude obtained by Kapitza (*14*) in Figure 7. The agreement is fairly good considering the experimental difficulties and it is somewhat better than Kapitza's predictions. However, it appears that the experimental values increase with the Weber number. Since the mathematical rigor of this method is uncertain we cannot draw any conclusion.

STREAMLINES

The present development has produced an improved and qualitatively satisfying description of fluid motion in wavy films. It would be of great interest to use this model for prediction and interpretation of heat and mass transfer phenomena. For this purpose it is important to construct the fluid streamlines. For some wavy motions streamlines are known (*26*). To the authors' knowledge, they have never been determined mathematically for film flow. In his preliminary paper Kapitza sketches them qualitatively (*13*) and Portalski (*25*) uses his result. But an interesting feature of the present model is that it enables determination of the streamlines from the velocity profile given by Equations (17) and (18) without presupposing a surface shape. Streamlines are defined by

$$\frac{dx}{v_x} = \frac{dy}{v_y} \quad (70)$$

Using expressions (17) and (18) for v_x and v_y, and the subscript s for conditions along a streamline, we can put Equation (70) in the form

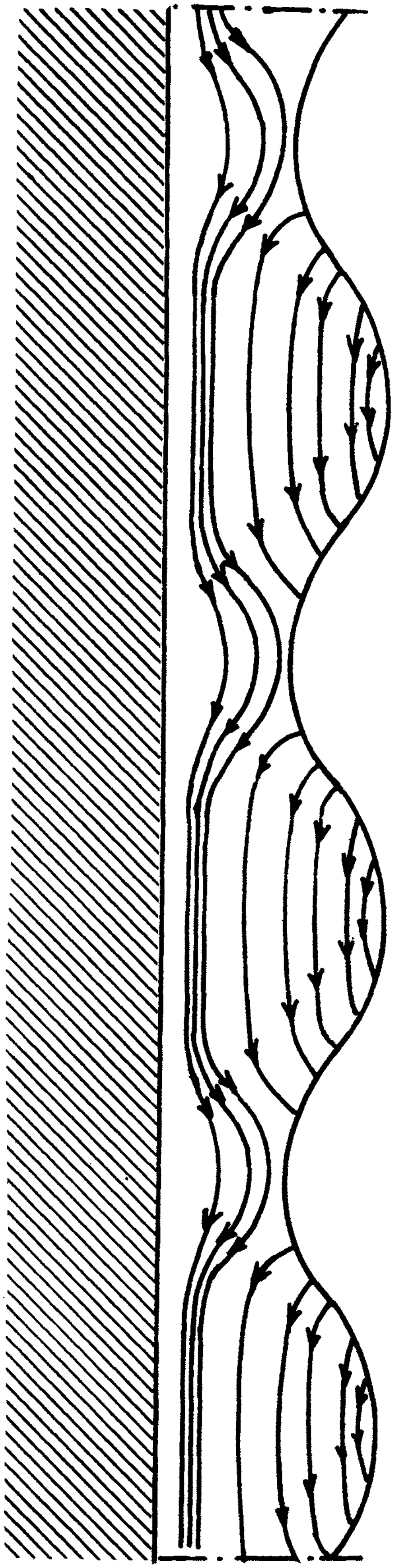

Fig. 8a. Instantaneous streamlines of a falling liquid film for $N_{cA} < 1$ showing a unidirectional flow. For an enlarged view, see Figure 8b.

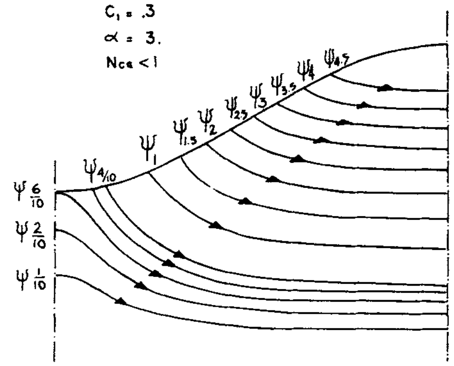

Fig. 8b. Enlarged view of Figure 8a, for one-half of a wavelength. Instantaneous streamlines of a falling liquid film for $N_{cA} < 1$ corresponding to unidirectional flow. Values of the stream function Ψ, positive in the whole field, are indicated.

$$y' \equiv \left(\frac{dy}{dx}\right)_s = \frac{v_y}{v_x} = -\frac{1}{v}\frac{\partial v}{\partial x}\frac{\left(\frac{y^2}{h^2}-\frac{y^3}{6h^3}\right)}{\left(\frac{y}{h}-\frac{y^2}{2h^2}\right)} + \frac{1}{h}\frac{\partial h}{\partial x}\frac{\left(\frac{y^2}{2h^2}-\frac{y^3}{3h^3}\right)}{\left(\frac{y}{h}-\frac{y^2}{2h^2}\right)} \quad (71)$$

By setting $\delta = \frac{y}{h}$, $\delta' = \left(\frac{d\delta}{dX}\right)_s$

and using the relation

$$y' = \delta' h + h'\delta \quad (72)$$

Equation (71) can be brought under the form

$$\frac{2\delta'}{\delta} + \frac{(3-\delta)'}{(3-\delta)} = -\frac{(vh)'}{(vh)} \quad (73)$$

This expression can be simply integrated to give

$$\bar{v}\, h\, \delta^2\, (3-\delta) = C_s \quad (74)$$

where C_s is a constant along *any given streamline*. It should be noted that Equation (74) gives an implicit relation between x and y, since v and h are functions of x. It is also clear that to each value of C_s corresponds a fixed value of the stream function Ψ which is also constant along a streamline. One now needs a relation between C_s and the stream function Ψ. Since by definition of Ψ

$$d\Psi = v_x dy - v_y dx \quad (75)$$

one can write

$$\frac{d\Psi}{dC_s} = \frac{d\Psi}{dy}\bigg|_x \bigg/ \frac{dC_s}{dy}\bigg|_x = \frac{v_x}{6v\frac{y}{h} - 3v\frac{y^2}{h^2}} = \frac{1}{2} \quad (76)$$

Since $\Psi = C_s = 0$ at the wall which is a streamline

$$\Psi = \frac{1}{2} C_s \quad (77)$$

Insertion of expressions (6) and (22) in (74) yields

$$\Psi = \frac{3}{2}\delta^2 (3-\delta)\left(1 + \alpha\, C_1 \sin\frac{2\pi x}{\lambda}\right)$$

The equation of the streamline is

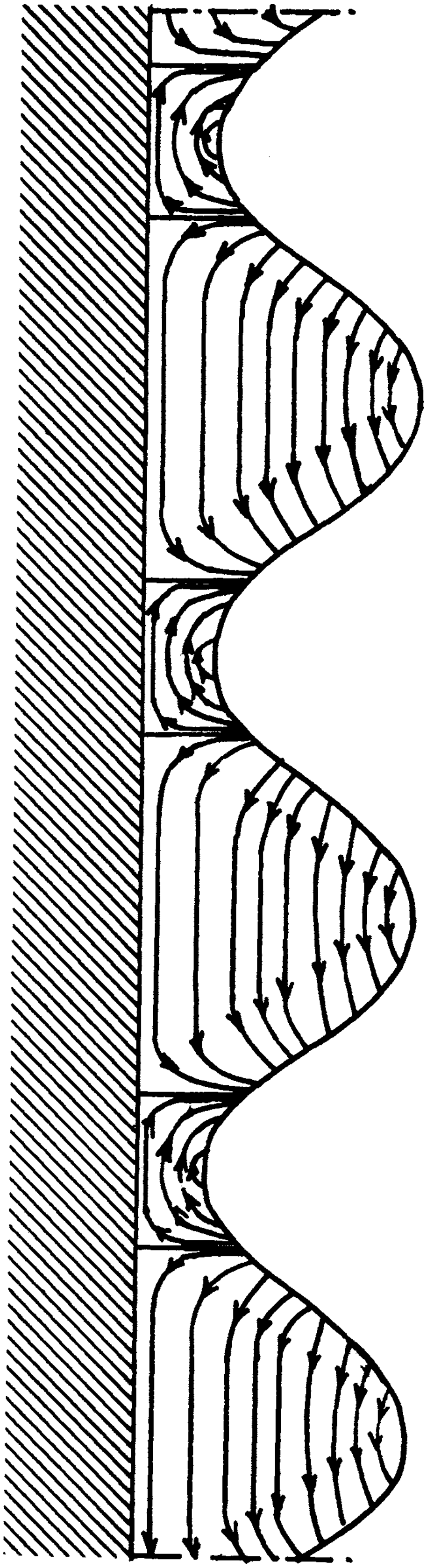

Fig. 9a. Instantaneous streamlines of a falling liquid film for $N_{CA} > 1$, showing a flow reversal. For an enlarged view, see Figure 9b.

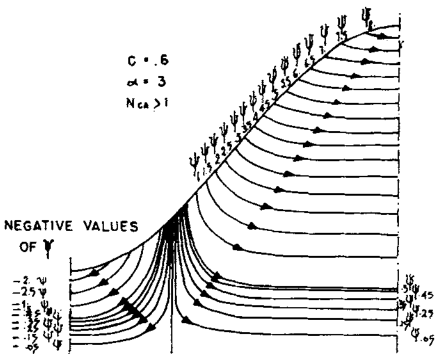

Fig. 9b. Enlarged view of Figure 9a. for one-half of a wavelength of instantaneous streamlines of a falling liquid film for $N_{CA} > 1$ corresponding to flow reversal. The values of the stream function Ψ are indicated. Negative values of Ψ appear in the zone of reversed flow.

$$x = \frac{\lambda}{2\pi} \text{arc sin} \frac{1}{\alpha C_1} \left(\frac{\Psi}{\frac{3}{2} \delta^2 (3 - \delta)} - 1 \right) \quad (78)$$

when

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

and

$$\pi - x = \frac{\lambda}{2\pi} \text{arc sin} \frac{1}{\alpha C_1} \left(\frac{2\Psi}{3 \delta^2 (3 - \delta)} - 1 \right) \quad (79)$$

when

$$\frac{\pi}{2} < x < \frac{3\pi}{2}$$

This equation represents the streamline existing in the fluid at a given time. Representative examples are shown in Figures 8 and 9.

Two families of streamlines appear: *unidirectional streamlines,* corresponding to slightly deformed parallel flow, and to positive values only of the stream function; and *reversal flow streamlines,* in which coexist streamlines corresponding to positive and negative values of the stream function, suggesting a swirl in the liquid.

The transition between these two types of flow is easily determined. The flow reversal appears as soon as the average velocity $\bar{v}$ can vanish. From Equation (22) it can be seen that this phenomenon arises when

$$c\phi + v_0 = 0 \quad (80)$$

or

$$\alpha\phi + 1 = 0 \quad (81)$$

which can be written

$$\alpha C_1 \sin \frac{2\pi x}{\lambda} = -1 \quad (82)$$

The determining parameter for flow reversal appears to be the dimensionless celerity-amplitude number, N_{ca}

$N_{ca} > 1$	unidirectional flow
$N_{ca} > 1$	flow reversal

These two types of flow can be expected to reveal very different mechanisms of surface renewal.

Further work is under way to relate quantitatively these preliminary results to transport behavior.

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NOTATION

A defined by $N_\lambda = A\, N_{Re}^{-1/2}$. Values of A are given in Table 1, dimensionless
c = wave celerity or phase velocity of the wave, cm./sec.
C_s = constant along a streamline, sec.$^{-1}$
C_1 = dimensionless wave amplitude
C_{1m} = C_1 corresponding to the minimum of $\bar{N}_\Phi{}^\lambda$
g = acceleration due to gravity, cm./sec.^{2}
$h(x, t)$ = surface coordinate, cm.
h_0 = film thickness averaged over a wavelength $= \frac{1}{\lambda}\int_0^\lambda h\,dx$, cm.
N_{ca} = celerity-amplitude number $= \alpha\, C_1$, dimensionless
N_{cw} = dimensionless complete wave number $= (2\pi h_0 C_1)/\lambda$
N_{dw} = dynamic wave number $= (2\pi h_0 C_1 \alpha)/\lambda$, dimensionless
N_{Fr} = Froude number $= v_0/\sqrt{g h_0} = \frac{g^{1/2} h_0^{3/2}}{3\nu} = \frac{N_{Re}^{1/2}}{\sqrt{12}}$, dimensionless
N_{fp} = fluid properties number $= \left(3\frac{\gamma^3}{g\nu^4}\right)^{1/5} = \left(\frac{3\rho\sigma^3}{g\mu^4}\right)^{1/5}$
N_{Re} = Reynolds number $= 4Q/\nu = 4\, N_{fp}\, N_{We}^{3/5}$, dimensionless
N_{v_x} $= \frac{2}{3} v_x/v_0$, dimensionless
N_{v_y} $= \frac{2}{3} v_y/v_0$, dimensionless
N_W = dimensionless wave number $= (2\pi h_0)/\lambda$
N_{We} = Weber number $= (v_0^2 h_0)/\gamma = (g^2 h_0^5)/(9\nu^2\gamma)$, dimensionless
N_Φ $= (\Phi_v)/(\bar{\Phi}_{VN})$, dimensionless
$\bar{N}_\Phi{}^\lambda$ $= \frac{1}{\lambda}\int_0^\lambda N_\Phi dx$, dimensionless
N_λ = dimensionless wavelength $= \frac{\lambda}{2\pi}\sqrt{g/\gamma}$
Q = flow rate $= v_0 h_0$, sq. cm./sec.
t = time, sec.
v_x = x component of the velocity, cm./sec.
v_y = y component of the velocity, cm./sec.
v_0 = average velocity for a cross section of width, h_0, cm./sec.
$\bar{v}$ = average velocity over a cross section $= \frac{1}{h}\int_0^h v_x dy$, cm./sec.
X $= \frac{2\pi x}{\lambda}$, dimensionless
x = Cartesian coordinate parallel to plane in direction of main flow, cm.
y = Cartesian coordinate perpendicular to plane, cm.

Greek Letters

α = dimensionless wave celerity $= c/v_0$
γ = kinematic surface tension $= \sigma/\rho$, cc./sec.^{2}
δ = dimensionless transverse coordinate for streamline calculation $= y/h$
λ = wavelength
μ = dynamic viscosity, g./(cm.)(sec.)
ν = kinematic viscosity $= \mu/\rho$, sq. cm./sec.
ρ = density, g./cc.
σ = dynamic surface tension, dyne/cm.
ϕ = free surface deformation function of flowing film $= \phi(x, t)$, dimensionless
Φ_v = viscous dissipation function, sec.$^{-2}$
$\bar{\Phi}_{VN}$ $= 3\, v_0^2/h_0$, cm./sec.^{2}
Ψ = stream function, sec.$^{-1}$

Subscripts

0 = at $x = 0$
s = along a streamline

Superscript

λ = integrated over the wavelength

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